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On the Moore Formula of Compact Nilmanifolds

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Abstract. Let G be a connected and simply connected two-step nilpotent Lie group and Γ a lattice subgroup of G. In this note, we give a new multiplicity formula, according to the sense of Moore, of irreducible unitary representations involved in the decomposition of the quasi-regular representation $\mathrm{Ind}_\Gamma^G(1)$. Extending then the Abelian case.

Key words: nilpotent Lie group; lattice subgroup; rational structure; unitary representation; Kirillov theory

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1 Introduction

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and suppose G contains a discrete cocompact subgroup Γ. Let $\mathsf{R}_\Gamma = \mathrm{Ind}_\Gamma^G(1)$ be the quasi-regular representation of G induced from Γ. Then R_Γ is direct sum of irreducible unitary representations each occurring with finite multiplicity [3]; we will write

$$\mathsf{R}_\Gamma = \sum_{\pi\in(G:\Gamma)} \mathbf{m}(\pi, G, \Gamma, 1)\pi.$$

A basic problem in representation theory is to determine the spectrum $(G : \Gamma)$ and the multiplicity function $\mathbf{m}(\pi, G, \Gamma, 1)$. C.C. Moore first studied this problem in [7]. More precisely, we have the following theorem.

Theorem 1. *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and Γ a lattice subgroup of G (i.e., Γ is a discrete cocompact subgroup of G and $\log(\Gamma)$ is an additive subgroup of $\mathfrak{g}$). Let π be an irreducible unitary representation with coadjoint orbit $\mathcal{O}_\pi^G$. Then π belongs to $(G : \Gamma)$ if and only if $\mathcal{O}_\pi^G$ meets $\mathfrak{g}_\Gamma^* = \{l \in \mathfrak{g}^*,\ \langle l, \log(\Gamma)\rangle \subset \mathbb{Z}\}$ where $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$.*

Later R. Howe [4] and L. Richardson [12] gave independently the decomposition of R_Γ for an arbitrary compact nilmanifold. In this paper, we pay attention to the question wether the multiplicity formula

$$\mathbf{m}(\pi, G, \Gamma, 1) = \#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^*/\Gamma] \qquad \forall\, \pi \in (G : \Gamma)$$

required in the Abelian context, still holds for non commutative nilpotent Lie groups (we write $\#A$ to denote the cardinal number of a set A). In [7], Moore showed the following inequality

$$\mathbf{m}(\pi, G, \Gamma, 1) \leq \#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^*/\Gamma] \qquad \forall\, \pi \in (G : \Gamma), \tag{1}$$

where Γ is a lattice subgroup of G, and produced an example for which the inequality (1) is strict. More precisely, he showed that

$$\mathbf{m}(\pi, G, \Gamma, 1)^2 = \#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^*/\Gamma] \qquad \forall\, \pi \in (G : \Gamma) \tag{2}$$

in the case of the 3-dimensional Heisenberg group and Γ a lattice subgroup. The present paper aims to show that every connected, simply connected two-step nilpotent Lie group satisfies equation (2). We present therefore a counter example for 3-step nilpotent Lie groups.

2 Rational structures and uniform subgroups

In this section, we summarize facts concerning rational structures and uniform subgroups in a connected, simply connected nilpotent Lie groups. We recommend [2] and [9] as a references.

2.1 Rational structures

Let G be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a *rational structure* if there is a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $\{X_1, \ldots, X_n\}$ with rational structure constants.

Let $\mathfrak{g}$ have a fixed rational structure given by $\mathfrak{g}_{\mathbb{Q}}$ and let $\mathfrak{h}$ be an $\mathbb{R}$-subspace of $\mathfrak{g}$. Define $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$. We say that $\mathfrak{h}$ is *rational* if $\mathfrak{h} = \mathbb{R}\text{-span}\{\mathfrak{h}_{\mathbb{Q}}\}$, and that a connected, closed subgroup H of G is *rational* if its Lie algebra $\mathfrak{h}$ is rational. The elements of $\mathfrak{g}_{\mathbb{Q}}$ (or $G_{\mathbb{Q}} = \exp(\mathfrak{g}_{\mathbb{Q}})$) are called *rational elements* (or *rational points*) of $\mathfrak{g}$ (or G).

2.2 Uniform subgroups

A discrete subgroup Γ is called *uniform* in G if the quotient space G/Γ is compact. The homogeneous space G/Γ is called a *compact nilmanifold*. A proof of the next result can be found in Theorem 7 of [5] or in Theorem 2.12 of [11].

Theorem 2 (the Malcev rationality criterion)**.** *Let G be a simply connected nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then G admits a uniform subgroup Γ if and only if $\mathfrak{g}$ admits a basis $\{X_1, \ldots, X_n\}$ such that*

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k, \qquad \forall\, 1 \leq i, j \leq n,$$

where the constants c_{ijk} are all rational. (The c_{ijk} are called the structure constants of $\mathfrak{g}$ relative to the basis $\{X_1, \ldots, X_n\}$.)

More precisely, we have, if G has a uniform subgroup Γ, then $\mathfrak{g}$ (hence G) has a rational structure such that $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}\text{-span}\{\log(\Gamma)\}$. Conversely, if $\mathfrak{g}$ has a rational structure given by some $\mathbb{Q}$-algebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$, then G has a uniform subgroup Γ such that $\log(\Gamma) \subset \mathfrak{g}_{\mathbb{Q}}$ (see [2] and [5]). If we endow G with the rational structure induced by a uniform subgroup Γ and if H is a Lie subgroup of G, then H is rational if and only if $H \cap \Gamma$ is a uniform subgroup of H. Note that the notion of rational depends on Γ.

2.3 Weak and strong Malcev basis

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathscr{B} = \{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$. We say that $\mathscr{B}$ is a weak (resp. strong) Malcev basis for $\mathfrak{g}$ if $\mathfrak{g}_i = \mathbb{R}\text{-span}\{X_1, \ldots, X_i\}$ is a subalgebras (resp. an ideal) of $\mathfrak{g}$ for each $1 \leq i \leq n$ (see [2]).

Let Γ be a uniform subgroup of G. A strong or weak Malcev basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ is said to be *strongly based on* Γ if

$$\Gamma = \exp(\mathbb{Z}X_1) \cdots \exp(\mathbb{Z}X_n).$$

Such a basis always exists (see [5, 2, 6]).

A proof of the next result can be found in Proposition 5.3.2 of [2].

Proposition 1. *Let Γ be uniform subgroup in a nilpotent Lie group G, and let $H_1 \subsetneqq H_2 \subsetneqq \cdots \subsetneqq H_k = G$ be rational Lie subgroups of G. Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_{k-1}, \mathfrak{h}_k = \mathfrak{g}$ be the corresponding Lie algebras. Then there exists a weak Malcev basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{h}_1, \ldots, \mathfrak{h}_{k-1}$. If the H_j are all normal, the basis can be chosen to be a strong Malcev basis.*

2.4 Lattice subgroups

Definition 1 ([7])**.** Let Γ be a uniform subgroup of a simply connected nilpotent Lie group G, we say that Γ is a lattice subgroup of G if $\log(\Gamma)$ is an Abelian subgroup of $\mathfrak{g}$.

In [7], Moore shows that if a simply connected nilpotent Lie group G satisfies the Malcev rationality criterion, then G admits a lattice subgroup.

We close this section with the following proposition [1, Lemma 3.9].

Proposition 2. *If Γ is a lattice subgroup of a simply connected nilpotent Lie group $G = \exp(\mathfrak{g})$ and $\{X_1, \ldots, X_n\}$ is a weak Malcev basis of $\mathfrak{g}$ strongly based on Γ, then $\{X_1, \ldots, X_n\}$ is a $\mathbb{Z}$-basis for the additive lattice $\log(\Gamma)$ in $\mathfrak{g}$.*

3 Main result

We begin with the following definition.

Definition 2. Let G be a connected, simply connected nilpotent Lie group which satisfies the Malcev rationality criterion, and let $\mathfrak{g}$ be its Lie algebra.

(1) We say that G satisfies the Moore formula at a lattice subgroup Γ if we have

$$\mathbf{m}(\pi, G, \Gamma, 1)^2 = \#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^* / \Gamma], \qquad \forall\, \pi \in (G : \Gamma)).$$

(2) We say that G satisfies the Moore formula if G satisfies the Moore formula at every lattice subgroup Γ of G.

Examples.

(1) Every Abelian Lie group satisfies the Moore formula.

(2) The 3-dimensional Heisenberg group satisfies the Moore formula (see [7, p. 155]).

The main result of this paper is the following theorem.

Theorem 3. *Every connected, simply connected two-step nilpotent Lie group satisfies the Moore formula.*

Before proving Theorem 3, we must review more of the Corwin–Greenleaf multiplicity formula.

3.1 The Corwin–Greenleaf multiplicity formula

Using the Poisson summation and Selberg trace formulas, L. Corwin and F.P. Greenleaf [1] gave a formula for $\mathbf{m}(\pi, G, \Gamma, 1)$ that depended only on the coadjoint orbit in $\mathfrak{g}^*$ corresponding to π via Kirillov theory. We state their formula for lattice subgroups. Let Γ be a lattice subgroup of a connected, simply connected nilpotent Lie group $G = \exp(\mathfrak{g})$. Let

$$\mathfrak{g}^*_\Gamma = \{l \in \mathfrak{g}^* : \langle l, \log(\Gamma)\rangle \subset \mathbb{Z}\}.$$

Let π_l be an irreducible unitary representation of G with coadjoint orbit $\mathcal{O}^G_{\pi_l} \subset \mathfrak{g}^*$ such that $\mathcal{O}^G_{\pi_l} \neq \{l\}$. According to Theorem 1, we have $\mathbf{m}(\pi_l, G, \Gamma, 1) > 0$ if and only if $\mathcal{O}^G_{\pi_l} \cap \mathfrak{g}^*_\Gamma \neq \emptyset$, so we will suppose this intersection is nonempty. The set $\mathcal{O}^G_{\pi_l} \cap \mathfrak{g}^*_\Gamma$ is Γ-invariant. For such Γ-orbit $\Omega \subset \mathcal{O}^G_{\pi_l} \cap \mathfrak{g}^*_\Gamma$ one can associate a number $c(\Omega)$ as follows: let $f \in \Omega$ and $\mathfrak{g}(f) = \ker(B_f)$, where B_f is the skew-symmetric bilinear form on $\mathfrak{g}$ given by

$$B_f(X, Y) = \langle f, [X, Y]\rangle, \qquad X, Y \in \mathfrak{g}.$$

Since $\langle f, \log(\Gamma)\rangle \subset \mathbb{Z}$ then $\mathfrak{g}(f)$ is a rational subalgebra. There exists a weak Malcev basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{g}(f)$ (see [2, Proposition 5.3.2]). We write $\mathfrak{g}(f) = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_s\}$. Let

$$A_f = \mathrm{Mat}\big(\langle f, [X_i, X_j]\rangle : \ s < i, j \leq n\big). \tag{3}$$

Then $\det(A_f)$ is independent of the basis satisfying the above conditions and depends only on the Γ-orbit Ω. Set

$$c(\Omega) = \big(\det(A_f)\big)^{-\frac{1}{2}}.$$

Then $c(\Omega)$ is a positive rational number and the multiplicity formula of Corwin–Greenleaf is

$$\mathbf{m}(\pi_l, G, \Gamma, 1) = \begin{cases} 1, & \text{if } \ \mathfrak{g}(l) = \mathfrak{g}, \\ \displaystyle\sum_{\Omega \in [\mathcal{O}^G_{\pi_l} \cap \mathfrak{g}^*_\Gamma / \Gamma]} c(\Omega), & \text{otherwise.} \end{cases} \tag{4}$$

For details see [1].

Proof of Theorem 3. Let $l \in \mathcal{O}^G_\pi \cap \mathfrak{g}^*_\Gamma$. The result is obvious if $\mathfrak{g}(l) = \mathfrak{g}$. Next, we suppose that $\mathfrak{g}(l) \neq \mathfrak{g}$. Since G is two-step nilpotent Lie group then $\mathfrak{g}(l)$ is an ideal of $\mathfrak{g}$, and hence we have $\mathfrak{g}(l) = \mathfrak{g}(f)$ for every $f \in \mathcal{O}^G_\pi$ and $\mathcal{O}^G_\pi = l + \mathfrak{g}(l)^\perp$ (see [2, Theorem 3.2.3]). On the other hand, as l belongs to $\mathfrak{g}^*_\Gamma$ then $\mathfrak{g}(l)$ is rational. By Proposition 5.3.2 of [2] there exists a Jordan–Hölder basis $\mathscr{B} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{g}(l)$. Set $\mathfrak{g}(l) = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_s\}$.

Then, for every $\Omega \in [\mathcal{O}^G_\pi \cap \mathfrak{g}^*_\Gamma / \Gamma]$ and for every $f \in \Omega$, we have

$$c(\Omega) = \det(A_f)^{-\frac{1}{2}} = \det(A_l)^{-\frac{1}{2}} = c(\Gamma \cdot l),$$

since $f|_{[\mathfrak{g},\mathfrak{g}]} = l|_{[\mathfrak{g},\mathfrak{g}]}$. It follows from (4) that

$$\mathbf{m}(\pi, G, \Gamma, 1) = \#[\mathcal{O}^G_\pi \cap \mathfrak{g}^*_\Gamma / \Gamma]\ c(\Gamma \cdot l). \tag{5}$$

Next, we calculate $\#[\mathcal{O}^G_\pi \cap \mathfrak{g}^*_\Gamma / \Gamma]$. Let $(t_1, \ldots, t_n) \in \mathbb{Z}^n$ and $f \in \mathcal{O}^G_\pi \cap \mathfrak{g}^*_\Gamma$. We have

$$\begin{aligned} \big(\exp(-t_1 X_1) \cdots \exp(-t_n X_n)\big) \cdot f &= f + \sum_{i=s+1}^{n} \left(\sum_{j=s+1}^{n} t_j \langle f, [X_j, X_i]\rangle \right) X_i^* \\ &= f + \sum_{i=s+1}^{n} \left(\sum_{j=s+1}^{n} t_j \langle l, [X_j, X_i]\rangle \right) X_i^*, \end{aligned}$$

since $f|_{[\mathfrak{g},\mathfrak{g}]} = l|_{[\mathfrak{g},\mathfrak{g}]}$. It follows that

$$\Gamma \cdot f = f + \sum_{j=s+1}^{n} \mathbb{Z} e_j,$$

where

$$e_j = \sum_{i=s+1}^{n} \langle l, [X_j, X_i] \rangle X_i^*, \qquad \forall\, s < j \leq n.$$

Let

$$\mathfrak{L} = \mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^* - f = \bigoplus_{s<i\leq n} \mathbb{Z} X_i^* \qquad \text{and} \qquad \mathfrak{L}_0 = \sum_{j=s+1}^{n} \mathbb{Z} e_j.$$

Since $\mathfrak{g}(l) \cap \mathbb{R}\text{-span}\,\{X_{s+1}, \ldots, X_n\} = \{0\}$, then the vectors $e_{s+1}, \ldots, e_n$ are linearly independent. Therefore, $\mathfrak{L}_0$ is a sublattice of $\mathfrak{L}$. It is well known that there exist $\varepsilon_{s+1}, \ldots, \varepsilon_n$ a linearly independent vectors of $\mathfrak{g}^*$ and $d_{s+1}, \ldots, d_n \in \mathbb{N}^*$ such that

$$\mathfrak{L} = \bigoplus_{s<i\leq n} \mathbb{Z} \varepsilon_i \qquad \text{and} \qquad \mathfrak{L}_0 = \bigoplus_{s<i\leq n} d_i \mathbb{Z} \varepsilon_i.$$

Consequently, we have

$$\#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^* / \Gamma] = d_{s+1} \cdots d_n.$$

Let $[\varepsilon_{s+1}, \ldots, \varepsilon_n]$ be the matrix with column vectors $\varepsilon_{s+1}, \ldots, \varepsilon_n$ expressed in the basis $(X_{s+1}^*, \ldots, X_n^*)$. From

$$\mathfrak{L} = \bigoplus_{s<i\leq n} \mathbb{Z} X_i^* = \bigoplus_{s<i\leq n} \mathbb{Z} \varepsilon_i,$$

we deduce that

$$[\varepsilon_{s+1}, \ldots, \varepsilon_n] \in \mathrm{GL}(n-s, \mathbb{Z}).$$

On the other hand, let $[e_{s+1}, \ldots, e_n]$ (resp. $[d_{s+1}\varepsilon_{s+1}, \ldots, d_n\varepsilon_n]$) be the matrix with column vectors $e_{s+1}, \ldots, e_n$ (resp. $d_{s+1}\varepsilon_{s+1}, \ldots, d_n\varepsilon_n$) expressed in the basis $(X_{s+1}^*, \ldots, X_n^*)$. Since

$$\mathfrak{L}_0 = \sum_{j=s+1}^{n} \mathbb{Z} e_j = \bigoplus_{s<i\leq n} d_i \mathbb{Z} \varepsilon_i,$$

then there exists $T \in \mathrm{GL}(n-s, \mathbb{Z})$ such that

$$[e_{s+1}, \ldots, e_n] = [d_{s+1}\varepsilon_{s+1}, \ldots, d_n\varepsilon_n] T.$$

The latter condition can be written

$${}^t A_l = [\varepsilon_{s+1}, \ldots, \varepsilon_n] \mathrm{diag}[d_{s+1}, \ldots, d_n] T.$$

Form this it follows that

$$\det(A_l) = d_{s+1} \cdots d_n.$$

Consequently

$$\#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^* / \Gamma] = \det(A_l). \tag{6}$$

Substituting the last expression (6) into (5), we obtain

$$\mathbf{m}(\pi, G, \Gamma, 1)^2 = \#[\mathcal{O}_\pi^G \cap \mathfrak{g}_\Gamma^* / \Gamma].$$

This completes the proof. ∎

As a consequence of the above result, we obtain the following result.

Corollary 1. *Let G be a connected, simply connected two-step nilpotent Lie group, let $\mathfrak{g}$ be the Lie algebra of G, and let Γ be a lattice subgroup of G. Let $l \in \mathfrak{g}^*$ such that the representation π_l appears in the decomposition of R_Γ. Let A_l as in* (3). *The multiplicity of π_l is*

$$\mathbf{m}(\pi_l, G, \Gamma, 1) = \begin{cases} 1, & \textit{if } \ \mathfrak{g}(l) = \mathfrak{g}, \\ (\det(A_l))^{\frac{1}{2}}, & \textit{otherwise.} \end{cases}$$

Remark 1. Note that in [10], H. Pesce obtained the above result more generally when Γ is a uniform subgroup of G.

4 Three-step example

In this section, we give an example of three-step nilpotent Lie group that does not satisfy the Moore formula. Consider the 4-dimensional three-step nilpotent Lie algebra

$$\mathfrak{g} = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_4\}$$

with Lie brackets given by

$$[X_4, X_i] = X_{i-1}, \qquad i = 2, 3,$$

and the non-defined brackets being equal to zero or obtained by antisymmetry. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. The group G is called the generic filiform nilpotent Lie group of dimension four. Let Γ be the lattice subgroup of G defined by

$$\Gamma = \exp(\mathbb{Z}X_1)\exp(\mathbb{Z}X_2)\exp(\mathbb{Z}X_3)\exp(6\mathbb{Z}X_4) = \exp(\mathbb{Z}X_1 \oplus \mathbb{Z}X_2 \oplus \mathbb{Z}X_3 \oplus 6\mathbb{Z}X_4).$$

Let $l = X_1^*$. It is clear that the ideal $\mathfrak{m} = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_3\}$ is a rational polarization at l. On the other hand, we have $\langle l, \mathfrak{m} \cap \log(\Gamma)\rangle \subset \mathbb{Z}$. Consequently, the representation π_l occurs in R_Γ (see [12, 4]). Now, we have to calculate $\#[\mathcal{O}_{\pi_l}^G \cap \mathfrak{g}_\Gamma^*/\Gamma]$.

Following [2] or [8], the coadjoint orbit of l has the form

$$\mathcal{O}_{\pi_l}^G = \left\{ X_1^* + tX_2^* + \frac{t^2}{2}X_3^* + sX_4^* : \ s, t \in \mathbb{R} \right\}.$$

On the other hand, it is easy to verify that

$$\mathfrak{g}_\Gamma^* = \mathbb{Z}\text{-span}\left\{ X_1^*, \ldots, X_3^*, \frac{1}{6}X_4^* \right\}.$$

Therefore

$$\mathcal{O}_{\pi_l}^G \cap \mathfrak{g}_\Gamma^* = \left\{ X_1^* + tX_2^* + \frac{t^2}{2}X_3^* + \frac{s}{6}X_4^* : \ s \in \mathbb{Z}, t \in 2\mathbb{Z} \right\}.$$

Let

$$f_{t_0,s_0} = X_1^* + t_0X_2^* + \frac{t_0^2}{2}X_3^* + \frac{s_0}{6}X_4^* \in \mathcal{O}_{\pi_l}^G \cap \mathfrak{g}_\Gamma^*$$

and

$$\gamma = \exp(rX_2)\exp(sX_3)\exp(6tX_4) \in \Gamma.$$

We calculate

$$\mathrm{Ad}^*(\gamma)f_{t_0,s_0} = X_1^* + (t_0 - 6t)X_2^* + \frac{(t_0 - 6t)^2}{2}X_3^* + \left(\frac{s_0}{6} + st_0 + r - 6st\right)X_4^*.$$

Then (see [8])

$$\begin{aligned}\mathrm{Ad}^*(\Gamma)f_{t_0,s_0} &= \left\{X_1^* + (t_0 + 6t)X_2^* + \frac{(t_0 + 6t)^2}{2}X_3^* + \left(\frac{s_0}{6} + s\right)X_4^* : \ s, t \in \mathbb{Z}\right\}\\ &= \{f_{t_0+6t,s_0+6s} : \ s, t \in \mathbb{Z}\}.\end{aligned}$$

From this we deduce that $\#[\mathcal{O}^G_{\pi_l} \cap \mathfrak{g}^*_\Gamma/\Gamma] = 3 \cdot 6 = 18$, and hence

$$\mathbf{m}(\pi_l, G, \Gamma, 1)^2 \neq \#[\mathcal{O}^G_{\pi_l} \cap \mathfrak{g}^*_\Gamma/\Gamma].$$

Therefore, the group G does not satisfy the Moore formula at Γ.

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